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Explicit solutions for a steady vortex–wave interaction

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A class of equilibria of the Euler equations is derived describing a vortex dipole of non-zero circulation interacting with the interface between a uniform shear layer and an irrotational region. The flow field and the shape of the deformed shear layer profile are given in terms of explicit mathematical formulae. Properties of the solutions are discussed. They share many qualitative features with a new translating quasi-geostrophic V-state solution recently found by McDonald (2004).

1. Introduction

In geophysical fluid dynamics there is much interest in finding stable self-propagating vortex–wave solutions where a localized vortex structure interacts with some interface of vorticity (or potential vorticity) associated with, for example, an escarpment or an interface separating two shear flows of piecewise-constant vorticity. One aim of such studies is understanding the existence of long-lived geophysical and astrophysical phenomena such as Jupiter's Great Red Spot and other well-defined vortical structures in zonal jet flows. Bell (1990) investigated a basic model where a point vortex interacts with a horizontal interface separating two regions of uniform potential vorticity. Related studies are by Dunn, McDonald & Johnson (2001) and McDonald (2002).

This paper presents a new class of equilibria of the Euler equations representing a steady interaction between a localized vortex structure and a vortical interface. This investigation has been motivated by the new travelling V-state solution recently found by McDonald (2004). McDonald's solution consists of two line vortices interacting with an interface representing a jump in potential vorticity caused by the presence of an escarpment. Since his solution involves combinations of vortex patches and point vortices and also turns out to have zero net circulation, McDonald intuited the possible relevance of a new analytical construction technique for vortical equilibria of the Euler equations originally developed in Crowdy (1999) (and subsequently generalized in a number of directions, e.g. Crowdy 2002*a*, *b*). This approach involves hybrid combinations of point vortices and vortex patches.

In similar spirit to McDonald's paper, the flow configuration considered here is a basic model of a geophysical flow. To motivate the configuration, it is first observed that the pair of point vortices in McDonald's solution essentially lends the embedded vortex a dipolar character possessing a non-zero net circulation. Second, it is known (e.g. Bell 1990; McDonald 2004) that the barotropic quasi-geostrophic equations with step topography are dynamically equivalent to the two-dimensional Euler equations with a piecewise-constant shear velocity playing the role of topography in providing a jump in the background vorticity distribution. Motivated by these two facts, the model

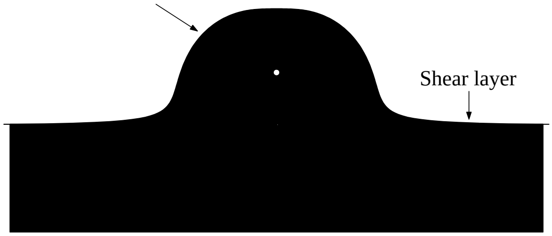

FIGURE 1. Schematic illustrating the vortex–wave. An infinite shear layer with uniform vorticity containing a vortex dipole with non-zero net circulation sits below a stagnant (irrotational) flow region. The configuration is assumed to be in steady equilibrium.

system to be considered here comprises a singular vortex dipole with non-zero net circulation sitting in a shear flow and interacting with a vorticity jump separating this shear flow region and an irrotational region. An attractive feature of the mathematical solutions is that they are completely explicit. In order to derive such solutions, it is necessary to assume that the upper region of irrotational flow is stagnant. While this idealization limits the physical relevance of the solutions, it is expected that more general solutions exist in which the flow above the interface is non-trivial. It is likely that such generalized solutions, which will be more physical, can be found using perturbation or numerical methods based on the class of explicit solutions derived herein.

2. Mathematical formulation

We seek solutions for a dipolar vortex with circulation near an infinite interface separating a region of irrotational flow from a uniform shear flow. The flow is assumed to be incompressible. As $x \to \pm\infty$, the interface is supposed to become flat and tend to $y=0$. Far from the interface, as $y \to -\infty$, the shear flow is assumed to have the form

$$(u, v) = (y, 0). \tag{2.1}$$

The vorticity in the region of shear is taken to equal -1 everywhere. It will be assumed that the dipolar vortex sits inside this rotational region. Let D denote the rotational region and ∂D the interface. Attention is restricted to a class of solutions in which the fluid above the interface is stagnant (and therefore trivially irrotational). Figure 1 shows a schematic of the flow configuration.

In terms of the standard complex variables $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$, the velocity field (u, v) we seek is of the complexified form

$$u - \mathrm{i}v = \begin{cases} \frac{1}{2}\mathrm{i}\,(\bar{z} - F(z)), & z \in D \\ 0, & z \notin D \end{cases} \tag{2.2}$$

where $F(z)$ must have a second-order pole at the point in space corresponding to the vortex dipole. The function $F(z)$, which is an analytic function of z (except for the isolated pole), represents the irrotational component of the velocity field. In order that this vortex has a net circulation, the residue of $F(z)$ at this point must not vanish. The condition that, as $y \to -\infty$, the velocity has the form (2.1) is equivalent to the requirement that

$$F(z) \to z + O(z^{-1}) \quad \text{as } z \to \infty. \tag{2.3}$$

There is a kinematic and dynamic boundary condition to be satisfied on the interface. The kinematic condition is that ∂D must be a streamline while the dynamic condition that the fluid pressures must be continuous across the vortex jump is known to be equivalent to continuity of velocity on ∂D (Saffman 1992). Continuity of velocity immediately implies, using (2.2), that

$$\bar{z} = F(z) \quad \text{on } \partial D. \tag{2.4}$$

Equivalently, in the terminology of Crowdy (1999), defining the Schwarz function $S(z)$ of the curve ∂D to be the unique function, analytic in a strip-like neighbourhood containing ∂D, that is equal to $\bar{z}$ everywhere on ∂D, then

$$F(z) = S(z). \tag{2.5}$$

It remains to satisfy the kinematic condition on the interface. However, this follows automatically since it is equivalent to ψ, the streamfunction associated with the incompressible flow, being constant on the interface and

$$\mathrm{d}\psi = \psi_z \mathrm{d}z + \psi_{\bar{z}} \mathrm{d}\bar{z} = \tfrac{1}{4}\left(\bar{z} - F(z)\right)\mathrm{d}z + \tfrac{1}{4}\left(z - \overline{F(z)}\right)\mathrm{d}\bar{z} = 0 \quad \text{on } \partial D \tag{2.6}$$

where the last equality follows from (2.4) and we have used $u - \mathrm{i}v = 2\mathrm{i}\psi_z$. $\overline{F(z)}$ denotes the complex conjugate of $F(z)$. Finally, there is an additional constraint on the solution (dictated by the Helmholtz vortex theorems – Saffman 1992) that the non-self-induced contribution to the velocity field at the vortex dipole position must vanish in order for the solution to be in consistent hydrodynamic equilibrium.

It follows that

$$\psi(z, \bar{z}) = \begin{cases} \dfrac{1}{4}\left[z\bar{z} - \displaystyle\int^{z} S(z')\,\mathrm{d}z' - \int^{\bar{z}} \bar{S}(z')\,\mathrm{d}z'\right], & z \in D \\ 0, & z \notin D \end{cases} \tag{2.7}$$

where the conjugate function $\bar{S}(z)$ is defined as $\bar{S}(z) = \overline{S(\bar{z})}$. The functional form of ψ is the same as the class of solutions presented in Crowdy (1999).

3. Conformal mapping

It remains to establish whether a domain D exists having an associated Schwarz function with the required properties. We have found that they do exist. As in Crowdy (1999), a convenient way to construct them is to consider a conformal mapping from a parametric ζ-plane. Let the interior of the unit ζ-circle map to the unbounded domain D with the circle $|\zeta| = 1$ mapping to the interface. Let $z(\zeta)$ be the conformal map. Since the interface is of infinite extent, $z(\zeta)$ must have a simple pole singularity on the unit circle. This will be taken to be at $\zeta = 1$. The vortex dipole is assumed to be situated on the imaginary axis and the displaced interface will be taken to be reflectionally symmetric about this axis. This means that we can expect the point $\zeta = -1$ to correspond to the point of the interface on the axis of symmetry and that complex-conjugate points ζ and $\bar{\zeta}$ in the pre-image plane will correspond to reflectionally symmetric points in the physical plane. Thus, if z_a corresponds to $z(\zeta_a)$ then $-\bar{z}_a$ will be given by $z(\bar{\zeta}_a)$ so that $z_a = z(\zeta_a) = -\overline{(-\bar{z}_a)} = -\bar{z}(\zeta_a)$ leading to the conclusion that

$$z(\zeta) = -\bar{z}(\zeta) \quad \text{for all } \zeta. \tag{3.1}$$

Introduce the class of conformal mappings given by

$$z(\zeta)=\mathrm{i}R\left(\frac{1}{\zeta-1}+a\zeta+b\zeta^2+c\right) \tag{3.2}$$

where, in order that the map satisfies (3.1), the parameters a, b, c and R are taken to be real. It is assumed that a, b, c and R can be found such that (3.2) is a one-to-one map from the interior of the unit ζ-circle to the domain D. Equation (3.2) will be shown to be the required class of conformal maps for appropriate choices of the parameters.

Now, the Schwarz function $S(z)$ is given by

$$S(z)=\bar{z}=\overline{z(\zeta)}=\bar{z}(\bar{\zeta})=\bar{z}(\zeta^{-1}) \tag{3.3}$$

where we have used the fact that $\bar{\zeta}=\zeta^{-1}$ on $|\zeta|=1$. On use of (3.2), we obtain

$$S(z(\zeta))=-\frac{\mathrm{i}R\zeta}{1-\zeta}-\frac{\mathrm{i}aR}{\zeta}-\frac{\mathrm{i}bR}{\zeta^2}-\mathrm{i}Rc \tag{3.4}$$

or, on rearrangement,

$$S(z(\zeta))=\mathrm{i}R\left(1+\frac{1}{\zeta-1}-\frac{a}{\zeta}-\frac{b}{\zeta^2}-c\right). \tag{3.5}$$

While (3.5) is valid on $|\zeta|=1$ it is also valid off this circle by analytic continuation. Using (3.2), (3.5) can be written as

$$S(z(\zeta))=z+\mathrm{i}R-\mathrm{i}aR\left(\zeta+\frac{1}{\zeta}\right)-\mathrm{i}bR\left(\zeta^2+\frac{1}{\zeta^2}\right)-2\mathrm{i}Rc. \tag{3.6}$$

where we have eliminated $\mathrm{i}R(\zeta-1)^{-1}$. Note that as $\zeta\to 1$, $z\to\infty$ and

$$S(z(\zeta))\to z+\mathrm{i}R(1-2a-2b-2c)+O((\zeta-1)). \tag{3.7}$$

Combining (2.5) and (3.7) it is seen that $F(z)$ satisfies the required far-field condition (2.3) provided that

$$R\left(a+b+c-\tfrac{1}{2}\right)=0, \tag{3.8}$$

which constitutes our first constraint on the parameters.

It is noted that (3.4) has a second-order pole at $\zeta=0$ which maps to a point inside D. Since $z(\zeta)$ is a one-to-one map this implies that $S(z)$ also has a second-order pole inside D. For general a, b and c, this pole of $F(z)$ will have non-zero residue and will correspond to the desired point vortex dipole with non-zero net circulation. Therefore, let $z_d=z(0)$ be the position of this dipole in D. Near z_d, the complex velocity field inside D will have the general form

$$u-\mathrm{i}v=\frac{\mu}{2\pi(z-z_d)^2}-\frac{\mathrm{i}\Gamma}{2\pi(z-z_d)}+V+O((z-z_d)) \tag{3.9}$$

for some constants μ, Γ and V. In order that the solution is a consistent steady solution of the Euler equation, the solution must be such that $V=0$ since V is the non-self-induced velocity at the dipole. Here μ has an interpretation as the vortex dipole strength while Γ is its net circulation.

Some straightforward (but somewhat lengthy) algebraic manipulations using the Taylor expansion of $z(\zeta)$ about $\zeta=0$ leads to the equations

$$\frac{\mu}{2\pi}=\frac{bR^3(a-1)^2}{2},\quad -\frac{\mathrm{i}\Gamma}{2\pi}=-\frac{\mathrm{i}R^2(a(a-1)+2b(b-1))}{2} \tag{3.10}$$

while the condition that $V = 0$ is found to be equivalent to

$$b^3 - 2b^2 - b(a^2 - 3a + 1) + (a - 1) = 0. \tag{3.11}$$

Equation (3.11) is a second constraint on the conformal mapping parameters.

We choose to normalize the class of solutions so that the vortex dipole is situated at $z = \mathrm{i}$. This corresponds to

$$z_d = \mathrm{i} = \mathrm{i}R(c - 1) \tag{3.12}$$

which is a third constraint on R, a, b and c. To impose that the interface tends to $y = 0$ as $x \to \pm\infty$ note that, as $\zeta \to 1$,

$$z \to z_M(\zeta) + O(\zeta - 1). \tag{3.13}$$

where

$$z_M(\zeta) = \frac{\mathrm{i}R}{\zeta - 1} + \mathrm{i}R(a + b + c). \tag{3.14}$$

is a Mobius map. The condition that $z_M(\zeta)$ takes $|\zeta| = 1$ to the line $y = 0$ is found to be the same as condition (3.8).

With a as parameter, (3.11) gives a cubic polynomial equation for b. Once b is determined, (3.8) and (3.12) provide equations for c and R. Since the equation for b in terms of a is a cubic polynomial, the solution can be written down, leading to a completely explicit class of solutions with parameters $b = b(a)$, $c = c(a)$ and $R = R(a)$ available as known functions of a. Indeed, by completing the cube and making an identification with the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, some algebra leads to the following formula for b:

$$b(a) = \tfrac{2}{3} + \delta(a)\cos\left(\tfrac{1}{3}\cos^{-1}(-4\beta(a)/\delta(a)^3)\right) \tag{3.15}$$

where

$$\beta(a) = -2a^2/3 + 3a - 61/27, \quad \delta(a) = 2(a^2 - 3a + 7/3)^{1/2}/\sqrt{3}. \tag{3.16}$$

The three possible solutions of the cubic (3.11) are encapsulated in the multivaluedness of the inverse cosine function. Of the three generally distinct solutions, only one has been found to yield a one-to-one conformal mapping and is therefore the only one that is physically admissible. Simple algebraic manipulations then lead to

$$c(a) = 1/2 - a - b(a), \quad R(a) = -(1/2 + a + b(a))^{-1}, \tag{3.17}$$

while the vortex dipole strength and circulation are

$$\mu(a) = \pi R^3(a)b(a)(a - 1)^2, \quad \Gamma(a) = \pi R^2(a)[a(a - 1) + 2b(a)(b(a) - 1)]. \tag{3.18}$$

The (complex) velocity field in the shear layer is also available as an explicit function of $\zeta, \bar{\zeta}$:

$$u - \mathrm{i}v = \frac{R(a)}{2}\left(\frac{1}{\bar{\zeta} - 1} + a\bar{\zeta} + b(a)\bar{\zeta}^2 - \frac{\zeta}{1 - \zeta} - \frac{a}{\zeta} - \frac{b(a)}{\zeta^2}\right). \tag{3.19}$$

Finally, we remark that (3.2) can be rearranged to form a cubic equation for ζ with coefficients depending on z and the conformal mapping parameters. The solution of this cubic can be written down (e.g. using Cardan's formula) leading to an explicit formula for the inverse map $\zeta(z)$. Thus, the complex velocity field (3.19) can, in principle, be written down explicitly in terms of z and $\bar{z}$; however the resulting expression is complicated. The conformal mapping approach provides a much more convenient parametrization of these non-trivial solutions.

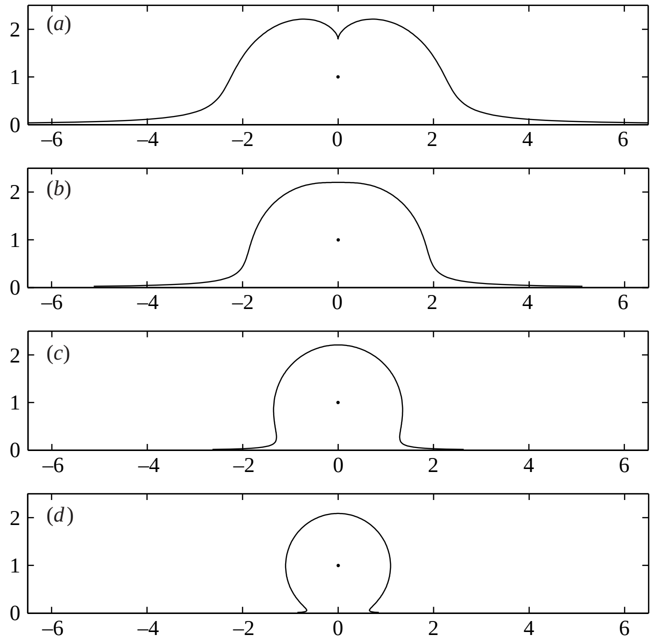

FIGURE 2. Equilibrium solutions for $a = a_{crit} = -0.9632$ and $a = -1.5, -3, -10$ ((a)–(d)). A black dot shows the position of the vortex dipole.

4. Characterization of the solutions

Solutions have been found to exist for all values of a in the interval

$$a \in (-\infty, a_{crit}] \tag{4.1}$$

where $a_{crit} = -0.9632$. Figure 2 shows the four representative interface profiles for $a = -0.9632, -1.5, -3$ and -10. The two limiting states $a = a_{crit}$ and $a \to -\infty$ are of interest. When $a = a_{crit} = -0.9632$ an inward-pointing cusp develops on the line of symmetry leading to a 'double hump' profile. Another allowable singularity is a 90° corner (see Overman 1986) and such limiting solutions on uniform vortex layers have been found by Broadbent & Moore (1985) in the case of uniformly travelling waves on a layer of finite depth. The solution here appears to be the first example of a cuspidal limiting state on a uniform vortex layer. It corresponds to a zero of z_ζ meeting the unit circle at $\zeta = -1$. The critical parameter $a_{crit} = -0.9632$ is the simultaneous solution of (3.11) and $z_\zeta(-1) = 0$, or

$$a - 2b(a) - 1/4 = 0. \tag{4.2}$$

As a gradually decreases, the profile becomes smoother and develops a single hump profile. The horizontal extent of the profile is greatest when a is close to a_{crit}. As $a \to -\infty$ the edges of the hump draw inwards so that the profile begins to adopt a near-circular shape. Figure 3 shows the profile in the case $a = -300$ with the circle $|z - \mathrm{i}| = 1$ superposed for comparison. The interface develops two points of very high curvature where the near-circular vortex sits on the near-flat shear layer.

Figure 4 shows graphs of the dipole strength μ and the circulation Γ as functions of a. Note first that μ is real so the dipole is always aligned with the horizontal axis. The circulation of the dipole is greatest in the critical double-hump state

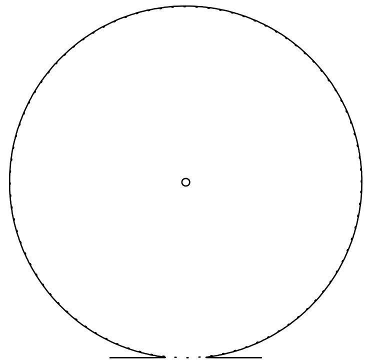

FIGURE 3. Solution for $a = -300$. The circle $|z - \mathrm{i}| = 1$ is superposed (dotted line) for comparison.

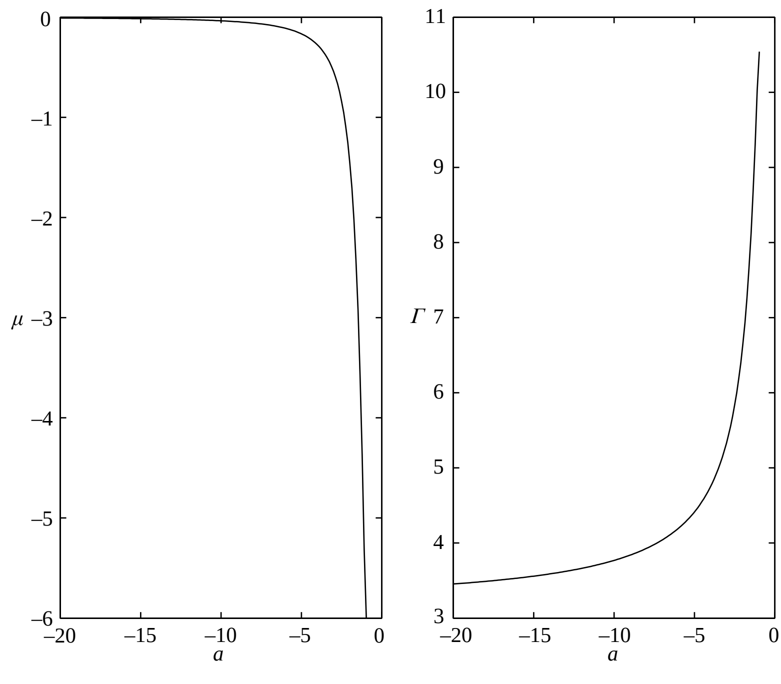

FIGURE 4. Graphs of dipole strength $\mu(a)$ and circulation $\Gamma(a)$ against a.

with $a = a_{crit}$. It is also in this state that the dipole strength μ has its greatest magnitude. On the other hand, as $a \to -\infty$, $\Gamma \to \pi$ while $\mu \to 0$. Since the profile in this limit tends to a near-circle with unit radius, the vortical configuration moves close to a situation where an area-π Rankine vortex of uniform vorticity -1 with a superposed point vortex at its centre of equal and opposite circulation π sits on top of a near-flat shear layer. Such a limiting configuration is consistent with ideas presented in Crowdy (2002*b*) concerning the possibility of constructing vortical equilibria with complicated geometry by superposing, and merging, shielded Rankine vortices. Indeed, the solutions here are 'smoothed-out' equilibria where a shielded Rankine vortex merges with a uniform shear layer.

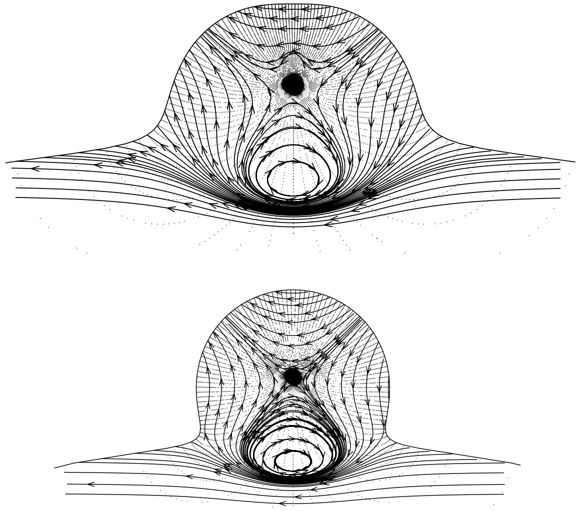

FIGURE 5. Streamlines for the solutions with $a=-1.5$ and $a=-3$. Note the presence of a large counter-rotating recirculation region below the vortex dipole.

McDonald (2004) finds a solution in which a pair of point vortices is situated below the height of an undisturbed vortical interface. In the solutions above, the vortex dipole is always above the far-field interface height and we have not found any solutions for vortex dipoles situated below it. However, the streamline patterns shown in figure 5 for $a=-1.5$ and -3 display interesting features. Most noticeable is the presence of a large counter-rotating recirculation region below the dipole. It is straightforward to show that there always exists a stagnation point on the imaginary axis just below the dipole. For a given a, the pre-image of such a stagnation point is a real solution for ζ between ± 1 of the nonlinear equation

$$\frac{1}{\zeta-1}+a\zeta+b(a)\zeta^2=\frac{\zeta}{1-\zeta}+\frac{a}{\zeta}+\frac{b(a)}{\zeta^2}. \tag{4.3}$$

The vertical position, as a function of a, of the stagnation point below the dipole is plotted in figure 6; it is clear that while the vortex singularity is above the far-field interface height, the centre of the region of fluid recirculating against the shear is always below it.

McDonald (2004) also finds that his equilibrium solution has zero net circulation. Consider the net circulation associated with the uniform vorticity of magnitude -1 which has crossed the undisturbed profile $y=0$ into $y>0$. The area that has crossed $y=0$ is given by the formula

$$\left|\frac{1}{2\mathrm{i}}\oint_{|\zeta|=1}\bar{z}(\zeta^{-1})z_\zeta(\zeta)\,\mathrm{d}\zeta-\frac{1}{2\mathrm{i}}\oint_{|\zeta|=1}\bar{z}_M(\zeta^{-1})z_{M\zeta}(\zeta)\,\mathrm{d}\zeta\right|. \tag{4.4}$$

An exercise in residue calculus shows that the area (4.4) is $\pi R(a)^2(a(a-1)+2b(a)(b(a)-1))$ which, by (3.18), equals $\Gamma(a)$. Hence, like the solution of McDonald (2004), the exact solutions share the property of having zero net circulation for all values of a.

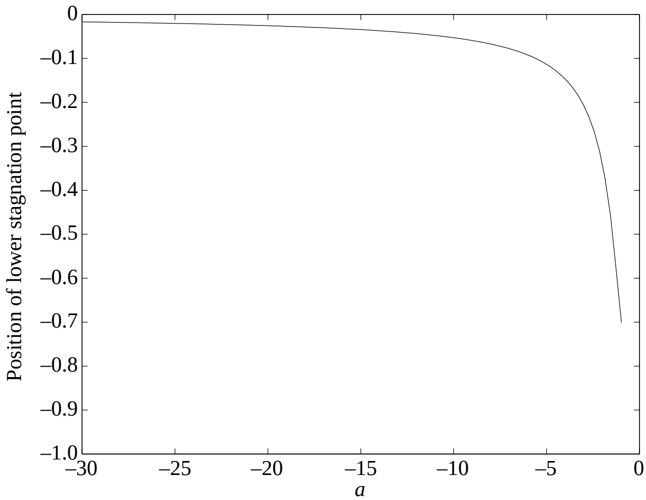

FIGURE 6. Height of the lower stagnation point (i.e. the centre of the lower recirculation region) as a function of a. It is always situated below the far-field height of the interface (which is at $y=0$).

5. Discussion

Significantly, there is no limit of the above solutions in which the interface tends to the linear flat state as the dipole strength and circulation vanish. The two limiting states are highly nonlinear structures, as are all the intermediate solutions. As such, we expect that they might be important in providing non-trivial initial states from which a numerical continuation procedure can be used to construct steadily translating equilibria not necessarily continuously connected to the flat state. Since they have a dipolar character, there may well exist continuations of these equilibria that are steadily translating. This is based on experience with the class of non-rotating multipolar equilibria derived in Crowdy (2002*a*) where it is found using numerical simulations that, when slightly perturbed, stable rotating structures can result. It may also be possible to use perturbation theory about the exact solutions to find a semi-analytic representation for generalized vortex–waves that have small non-zero translational velocities.

Modified solutions in which the point-vortex dipole is replaced by a point-vortex pair or desingularized to a uniform vortex patch are reasonable candidates for generalized equilibria. The exact solutions are likely to provide good initial estimates for the circulations of these modified models. There is also no finite value of a for which either μ or Γ becomes zero: both the circulation and dipolar character of the vortex are crucial for equilibrium. As mentioned earlier, in the solutions found here the dipole is always above the mean far-field height of the vortical interface. In the numerical algorithm of McDonald (2004), the two point vortices are constrained to be below the far-field interface height. The present results suggest that McDonald's solution class may well be generalizable if this constraint is relaxed.

Perhaps the most unphysical aspect of the solutions is the fact, observable in figure 5, that the streamlines intersect the vortical interface. This is a result of the assumption that the upper fluid is stagnant so that the vortical interface is a stagnation line where the local direction of the flow is not uniquely determined. We expect, however, that there exist generalized equilibria in which both the interface shape and dipole strength and circulation are 'close' to those of the exact solutions but where the motion of the upper fluid is non-trivial leading to a more physically realistic flow. Perturbation theory or numerical methods might be used to uncover such generalized solutions.

The solutions here share many qualitative features with a class of wave solutions on a layer of constant vorticity computed numerically in Vanden-Broeck (1994, 1995). He finds steadily translating solutions whose limiting forms consist of arbitrary numbers of circular Rankine vortices sitting on top of a region of uniform shear. This is reminiscent of the limiting solution here where a *shielded* Rankine vortex sits on top of a flat shear layer. This strongly suggests that there might exist a class of intermediate, steadily translating, vortex-wave solutions (not necessarily describable in exact mathematical form) where the circulation of the dipole differs from that of the above solutions. Vanden-Broeck (1995) also remarks on the fact that his solutions do not bifurcate from the uniform shear layer solution.

It should be possible to study the linear stability of the solutions as a function of a using complex-variable methods based on perturbed conformal maps as in Crowdy (2002*a*). The robustness of the structures to nonlinear perturbations could be examined using modified contour dynamics/surgery codes (Dritschel 1988).

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